#### **7. Economical interpretation of biological models**

**7.1. Introduction**

**Object** Evolution of economical firms

**Foundation** Known biological models

**Aim** Definition of the law of change in the firm capital using the analogy between biological and economical systems

**7.2. Malthus model**

Let us first consider the simplest economic system represented by unique firm. The process under study is characterized by the ***working capital*** of the firm *x*, which varies with time. It is assumed that the entire amount of money received from the sale of the manufactured goods is invested in production. The velocity of capital change is determined by the relation between the income and expenses of the company. Thus, we obtain the relation



where *A* and *B* characterize, respectively, the income and expense of the company per unit time. Their values, apparently, can be considered proportional to the values of capitals: the larger the capital of the company, the greater changes can occur over a fixed period of time. As a result, we obtain the relations *A=ax*, *B=bx*, where the positive constants *a* and *b* are process parameters and characterize the growth of the firm's income and expenses per unit time. Therefore, the considered process is described by differential equations

 **=*kx*, (1)

where the coefficient *k*=*a*–*b* is called ***capital gains***, it characterizes the change in the capital of the company per unit time and can take both positive and negative values. Equation (1) corresponding to the ***Malthus model*** is considered with the initial condition

 *х*(0) = *х*0, (2)

where the start-up capital *х*0 is the parameter of the problem, taking any positive values.

We know the properties of the Malthus model from biology; see Figure 1. It depends from the sign of the coefficient *k*. If this is positive, then we have an exponential growth of the function *x*. The function *x* is constant if *k=*0. Finally, it decreases and tends to zero if *k*<0.



Figure 1. Variants of the system evolution with different ratios between income and expense.

The results have the natural economical sense. For the positive values of the parameter *k*, the expense is less than the income. Hence, the firm obtains greater money and can product greater foods. It investes the money in production and obtain greater money again, so its capital is increasing. For the negative values of the coefficient *k*, the expense is greater than the income. The firm obtains a few moneys and decreases the production of food. Therefore, its capital is decreasing. Finally, if *k*=0, income coincide with expenses, and the production is constant.

 **7.3. Verhulst model**

For a more accurate description of the process under study, it is necessary to take into account restrictions on the consumption of manufactured goods. In this case, the capital gain is likely to decrease with increasing function *x*. Indeed, the greater the capital of the company, the greater the volume of output, and therefore, the greater the decline in demand for goods due to the inevitable saturation of the market. We accept the following dependence on *x* of capital gains *k*(*х*)*=а*(*D–qx*)–*b*, where *D* is the demand for manufactured products (an unchanged quantity of goods required per unit of time), *q* is the quantity of goods produced per unit of invested capital, *b* is the expenses of the company associated with the production of goods and not depending on the quantity of goods sold, and *a* is profit, received from the sale of a unit of goods. Thus, the income of the company is determined by the ratio between supply and demand. Thus, the state equation is

 *x*'(*t*) *=* [(*аD*–*b*)–*аqx*(*t*)] *x*(*t*). (3)

This is the Verhulst equation. It is considered with initial condition (2). The problem (3), (2) is called the ***Verhulst mod*e*l***.

We know the properties of the Verhulst model. The variants of the system evolution is shown in the Figure 2. If the inequality *аD*>*b* is true, then the income received from the sale of the goods exceeds the expenses of the company, for any initial state of the system (start-up capital) the capital of the company tends to *х*\*=(*D*–*b*/*а*)/*q* over time. If the starting capital of the company is small enough, i.e. the inequality *х*<*х*\* holds, then demand exceeds supply, the firm’s income grows, and it expands its production. However, as the market is saturated with goods, capital gains decrease. If the initial capital of the company is so large that the inequality *х*>*х*\* is satisfied, then there is an overproduction of goods (recall that in the framework of this model all money is invested in the production of a single product). Since not all manufactured products are bought up, the company suffers losses and gradually turns production down. In the course of restoring the balance between supply and demand, the decline in the capital of the company is reduced.



Figure 2. The firm capital tends to *х*\* if the consumption of goods is bounded.

**7.4. Cooperation model**

Now we consider the coexistence of two firms. There exists different forms of coexistence. First, we consider the case, when the firms mutually beneficial cooperate. Suppose that two firms produce components so that the products of one of the firms cannot be sold without the corresponding products of the other (roughly speaking, one firm produces bolts and the other produces nuts for these bolts). Under these conditions, firms simply cannot exist without each other. At the same time, the success of one of them favorably affects the financial situation of another company. As a result, we arrive at the equations

  (4)

where *хi* is the capital of the *i*-th company, *εi* is its ruin velocity in the absence of another company, *γi* is a parameter of the influence of the *j*-th company on the *i*-th, *i*=1,2, *j≠ i*. We know also the start capital of both firm *xi*0, so we have the initial conditions

 *xi*(0) = *xi*0, *i=*1,2. (5)

We have the ***cooperation model*** that coincides with simbiosis model.

The variants of the system evolution for the considered model is shown at the Figure 3. Suppose both initial states are small enough, then, the coefficients *ki* are negative, and the derivatives of both functions are negative, which means that the functions themselves decrease. Thus, in subsequent time instants, *ki* will be smaller. Consequently, derivatives will remain negative. Thus, over time, the functions will steadily decrease and tend to zero, i.e., to a stable equilibrium position Figure 3, curve 1. If, on the contrary, both initial values be so large that *ki* are positive, then the derivatives of both state functions are positive, which means that the functions themselves increase, so *ki* become larger. Thus, we observe their exponential growth; see curve 2. Suppose now the first function is large enough and the second on is small enough. Then the first function decreases, and the second increases. Further, two possible events are possible. Perhaps the function *x*1, decreasing, becomes smaller than the ratio , while the function *x*2, increasing, has not yet reached a value . Thus, both functions become small enough, so they tend to zero; see curve 3. However, another situation is also possible when the function *x*2, increasing, exceeds the value  earlier than the function *x*1, decreasing, reaches a value . Now both functions are large enough and increase unlimitedly; see curve 4. We have analogical situations with replacing of functions if initially the finction is small, and the second on is large; see curves 5 and 6.



Figure 3. Phase curves for the cooperation model.

 The obtained relations, up to the notation, coincide with the biological model of symbiosis (see the previous Chapter). Knowing the properties of the latter model, we can conclude that with small initial capital, both firms go bankrupt. Sufficiently large starting capitals of firms lead to complete prosperity, which is expressed in the unlimited increase in their capitals. If one of the companion firms is poor and the other rich, then for some time the rich company will gradually become poorer, supporting its weaker companion. At the same time, a poor company gets richer, having a sufficiently strong company as a companion. Then there are two possible scenarios. Perhaps the poor company will get stronger before the rich company is significantly weakened, and then both firms will grow rich indefinitely. However, the opposite situation is also possible when a stronger firm weakens before a weaker firm grows stronger. In this case, both firms will eventually go broke.

**7.5. Racketeer–entrepreneur model**

Consider another specific type of economic "cooperation". This is the relationship between the entrepreneur and the racketeer. The state functions here are their incomes *х*1 and *х*2, respectively. It is assumed that the entrepreneur has his own company with a constant source of income, and in the absence of a racketeer, his income increases with a velocity *ε*1. The racketeer lives solely at the expense of the entrepreneur, and in the absence of the latter spends the available funds with a velocity *ε*2. An increase in the entrepreneur's income leads to a proportional enrichment of the racketeer, which, in turn, is carried out through the withdrawal of funds from the entrepreneur. The process under study is described by a system of equations

  (6)

where the parameters *γ*1 and *γ*2 characterize the influence of the actors on each other. We add here the initial conditions (5). This is the racketeer–entrepreneur model that is the analogue of the predator–prey model.

The evolution of the system is described in Figure 4. The solutions of the problem are periodic functions here.



Figure 4. The system evolution for the racketeer–entrepreneur model.

 Give an interpretation of the results. With increasing entrepreneurial income, the racketeer has the opportunity to collect more and more tribute. However, as the "appetite" of the racketeer grows, the entrepreneur begins to go broke, which eventually affects the racketeer's income. Since the complete ruin of the entrepreneur is detrimental to the racketeer, he is forced to reduce the required sum of money. As a result, over time, the entrepreneur restores the financial position of his company, shaken due to exorbitant exactions. Then the racketeer gets the opportunity to catch up. A new process cycle begins.

**7.6. Economical competition model**

Now two firms are considered that produce the same product and are focused on the same consumer. It is assumed that in the absence of sales problems, both firms have some profit, i.e. the costs of manufacturing or purchasing products pay off as a result of their sale. All available capital is invested in production. Thus, having received additional funds from the sale of goods, firms expand production, and with a decrease in profits, the output of goods decreases accordingly. The system is described by the differential equations

  (7)

where *ai* characterizes the production efficiency, and *bi* characterizes the sales efficiency of manufactured products and is associated with the organization of advertising, service culture, etc. Thus, the higher the marketing efficiency of products, the less the problems arising with the sale of manufactured goods affect the rate of change of capital of the company. The equations (7) with standard initial conditions are called the economical competition model.

The variants of the system evolution for the competition model are given in Figure 5. It depends from the relation between ratios *ai*/*bi*. If *a*1/*b*1>*a*2/*b*2, then the system tends to the equilibrium state (*a*1/*b*1,0). For inverse case *a*1/*b*1<*a*2/*b*2, the system tends to another equilibrium state (0,*a*2/*b*2). Under the equality *a*1/*b*1=*a*2/*b*2, we can have many equilibrium positions that are the point of the line *k*1=*k*2=0 with non-negative coordinates. The concrete law of the system evolution depends from its initial state; see Figure 5.



Figure 5. Variants of the system evolution for the economical competition model.

Give the interpretation of the obtained results.For the qualitative behavior of the system, the decisive role here is the value *ai*/*bi*, which characterizes the effectiveness production and marketing of goods. A company whose production and marketing of goods is less efficient is inevitably ruined.

For *a*1/*b*1>*a*2/*b*2, the weaker second firm is ruined. In the case when the starting capitals of both are small enough, at first both firms completely sell their products, grow richer and expand their production. However, as the market is saturated with goods, the first firm gradually displaces its weaker competitor (curve 1). Its capital eventually stabilizes at a value equal to *a*1/*b*1, corresponding to the level of demand for manufactured products. If the starting capitals of both firms are too large, then the market is oversaturated with goods. Firms curtail production until the moment when the balance between supply and demand is restored (curve 2). After that, in a competitive environment, the stronger first firm wins. If initially the start-up capital of a more competitive company is too large and far exceeds the start-up capital of the second company, then in the process of reducing production a weak company goes bankrupt before there is a balance between supply and demand. In this case, the phase curve does not fall into the competition zone, which is a trapezoid bounded by the straight lines *k*1 = 0 and *k*2 = 0. Both state functions will monotonically decrease, the second of them tending to zero, and the first to a non-zero equilibrium position (curve 3). If the total starting capital of firms is not very small and not very large, then the initial state of the system is in the zone of competition. In this case, both state functions change monotonously, and the first of them increases and tends to its equilibrium position, and the second decreases to zero (curve 4).

If the inequality *a*1/*b*1<*a*2/*b*2 is true, then the second company wins in the process of competition. The behavior of the system in this case is similar to the previous one with the only difference that firms change places.

Equality *a*1/*b*1=*a*2/*b*2 means that firms are equally competitive. None of them are able to supplant their competitor. If the starting capitals of firms are small, then both of them expand their production until the market is saturated with goods (curves 1). If the goods on the market are in abundance, then both firms curtail production (curves 2). A feature of this option is the presence of an infinite number of equilibrium positions (segment 3). Moreover, the exit to a specific equilibrium position is determined by the initial state of the system.

**7.7. Niche model**

In the model considered above, for two firms producing the same product, long-term coexistence is impossible, since a weaker firm will inevitably go bankrupt. An exception here is the unlikely degenerate case of the equality *a*1/*b*1=*a*2/*b*2, when firms practically do not differ in properties. The situation will decisively change if each company focuses on its own customer, preferring the production of any specific product. At the same time, we get a model similar to the previously considered biological niche model

  (8)

where *ai* characterizes again the production efficiency *bij* is the sales efficiency of the *i*-th product by the *j*-th company. The equations (8) with the standard initial conditions are called the ***economic niche model***.

We know the variants of the system evolution for the analogical biological model. In this case, there are two comparison criteria that are *ai*/*bi*1 and *ai*/*bi*2. If *a*1/*b*11>*a*2/*b*21 and *a*1/*b*21>*a*2/*b*22, then the system tends to the point *A*1 with coordinates (*a*1/*b*11,0); see Figure 6a. *a*1/*b*11<*a*2/*b*21 and *a*1/*b*21<*a*2/*b*22, then the system tends to the point *B*2 with coordinates (0,*a*2/*b*22); see Figure 6b. For the case *a*1/*b*11<*a*2/*b*21 and *a*1/*b*21>*a*2/*b*22, two previous equilibrium positions can be realized. Depending on the initial state of the system, access to any of them is possible; see Figure 6c. Finally, we can have the inequalities *a*1/*b*11>*a*2/*b*21 and *a*1/*b*21<*a*2/*b*22. In this case, the system tends to the point *C* with coordinates (*x*1\*,*x*2\*) that are the solution to the system of algebraic equations



Besides, we can have the degenerate case with equalities *a*1/*b*11=*a*2/*b*21 and *a*1/*b*21=*a*2/*b*22. Under these conditions, the system tends to a point on the line  with non-negative coordinates. The concrete law of the system evolution depends of the initial state of the system.

Try to analyze these results. The outcome *a*) corresponds to the inequalities *a*1/*b*11>*a*2/*b*21 and *a*1/*b*21>*a*2/*b*22. Here, the first firm is superior to the second in both types of goods, since the product of the production and marketing efficiency of both types of goods is higher. As a result, the second firm will inevitably go bankrupt, i.e. its capital tends to zero. On outcome b), the opposite inequalities are realized *a*1/*b*11<*a*2/*b*21 and *a*1/*b*21<*a*2/*b*22. They correspond to the ruin of the first firm, which is weaker. Both of these variants actually return us to the previously considered model of economic competition. At the outcome *c*), the following relations hold *a*1/*b*11<*a*2/*b*21 and *a*1/*b*21>*a*2/*b*22. Here one of the firms survives, depending on the value of the initial state of the system. This situation corresponds to the case when the first company works more effectively with the second type of product, and the second with the first. The opposite case *d*), characterized by the inequalities *a*1/*b*11>*a*2/*b*21 and *a*1/*b*21<*a*2/*b*22 are more interesting.Here, any of the companies produce both types of goods, but give preference to their own. As a result, they are able to coexist peacefully, with each company filling its own economic niche. The results show that each company must certainly strive to find its consumer, otherwise it will be replaced by a stronger competitor. Only powerful firms, giant enterprises can afford the luxury of not being afraid of competition, confidently crowding out rivals from their sphere of production. The degenerate case corresponds to the equalities *a*1/*b*11=*a*2/*b*21 and *a*1/*b*21=*a*2/*b*22. This situation means that firms actually have the same properties, with the possible exception of the initial states. Under these conditions, none of them can displace the other, as a result of which both firms coexist. A similar situation was also observed in the competition model.

a) b)

c) d)

Figure 6. Variants of the system evolution for the niche model.

In all cases, with a fairly small start-up capital of both firms, a relatively small number of goods are produced. Demand exceeds supply, creating the opportunity for both firms to expand production. If, on the contrary, the starting capital of both firms is large enough, and the firms invest it in the production of goods, then supply in the market exceeds demand. Firms suffer losses and gradually curtail production. Only in the case when the capital of firms is neither very small nor very large, then events occur that depend on the efficiency of production and marketing of goods of both firms.

All economic models considered are similar to the corresponding biological models and, from a mathematical point of view, are no different from them.